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On the existence of antisymmetric or symmetric Lamb waves at nonlinear higher harmonics

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Abstract

This article theoretically studies the symmetry characteristics of Rayleigh–Lamb guided waves in nonlinear, isotropic plates. It has been known that the nonlinearity driven double harmonic in Lamb waves does not support antisymmetric motion. However, the proof of this has not been obvious. Moreover, little is known on nonlinearity driven Lamb harmonics higher than double. These gaps were here studied by the method of perturbation coupled with wavemode orthogonality and forced response. This reduced the nonlinear problem to a forced linear problem which was subsequently investigated to formulate an energy level constraint as the defining factor for the absence of antisymmetry at any order of higher harmonic. This constraint was then used to explain the reason behind the absence of antisymmetric Lamb waves at the double harmonic. Further, it was shown that antisymmetric motion is prohibited at all the higher order even harmonics, whereas all the higher order odd harmonics allow both symmetric and antisymmetric motions. Finally, experimental results corroborating theoretical conclusions are presented.

1. Introduction

The study of nonlinear elastic wave propagation has been of considerable interest for the last four decades. This, in part, is due to the fact that nonlinear parameters are, in general, more sensitive to structural defects than linear parameters [1]. Guided waves combine the sensitivity of nonlinear parameters with large inspection ranges [2]. Therefore, their application to nondestructive evaluation and structural health monitoring has drawn considerable research interest [3,4].

There are very few studies of guided nonlinear elastic waves due to the mathematical complexity of the problem. The nonlinear Navier equations are further complicated by geometrical constraints essential to the generation and sustenance of guided waves. A recent investigation pertaining to the second harmonic generation in guided Lamb waves was reported by Deng [5–8]. In these papers, the primary and secondary fields are represented by pairs of plane waves that satisfy stress free boundary conditions on the plate's surfaces. The authors conclude, among other points, that antisymmetric Lamb motion is not possible at the

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double harmonic. Since their analysis was not based upon modal decomposition, the formulation is complex and furthermore higher order nonlinearities were not addressed.

de Lima and Hamilton [9] and subsequently Deng [8] analyzed the problem of nonlinear guided waves in isotropic plates by using normal mode decomposition and forced response as suggested by Auld [10]. The authors used their formulation to explain the generation of the double harmonic and the cumulative growth of a phase-matched higher harmonic guided mode. However, their conclusions are limited to the double harmonics.

This paper starts with the same formulation adopted by de Lima and Hamilton to study the behavior of guided Lamb waves at higher harmonics. The formulation reduces a nonlinear problem to a forced linear problem. This approach has been used to break a problem consisting of several orders of nonlinearity to several forced problems, each corresponding to a single order of higher harmonic. It was found that the energy expression for a certain order of harmonic imposes constraints on the symmetry of the generated modes. This is used to prove that Rayleigh–Lamb antisymmetric motion is never allowed at even harmonics $(2\omega, 4\omega, 6\omega, \ldots)$, instead both symmetric and antisymmetric Rayleigh–Lamb motions can exist at the odd harmonics $(3\omega, 5\omega, 7\omega, \ldots)$.

2. Statement of the nonlinear problem

The equation of motion for nonlinear elasticity in a stress free plate (Fig. 1) is given by

$$(\lambda + 2\mu)\nabla(\nabla \cdot \mathbf{u}) - \mu\nabla \times (\nabla \times \mathbf{u}) + \mathbf{f} = \rho_0 \frac{\partial^2 \mathbf{u}}{\partial t^2} \tag{1}$$

with the stress free boundary condition:

$$(\mathbf{S^L} - \bar{\mathbf{S}}) \cdot \mathbf{n_y} = \mathbf{0} \quad \text{on } \mathscr{L} \tag{2}$$

where $\mathbf{u}$ is the particle displacement, λ and μ are Lamé constants, ρ_0 is the initial density of the body, y is the thickness direction, $\mathbf{n_y}$ is a unit vector perpendicular to the surface $\mathscr{L}$, $\mathbf{S_L}$ is the linear part of the Piola–Kirchoff stress tensor, and $\bar{\mathbf{S}}$ and $\mathbf{f}$ collect all nonlinear terms. For Rayleigh–Lamb modes considered here, $u_x = 0$ and $u_y, u_z \neq 0$.

Including first-order nonlinearity, Landau and Lifshitz proposed the following energy expression [11] which is expressed here in terms of displacement derivatives $u_{i,j}$:

$$\begin{aligned} E = {} & \tfrac{1}{4}\mu(u_{i,j} + u_{j,i})^2 + (\tfrac{1}{2}K - \tfrac{1}{3}\mu)(u_{i,i})^2 + (\mu + \tfrac{1}{4}A)(u_{i,j}u_{k,i}u_{k,j}) \\ & + (\tfrac{1}{2}K - \tfrac{1}{3}\mu + \tfrac{1}{2}B)[u_{i,i}(u_{j,k})^2] + \tfrac{1}{12}A(u_{i,j}u_{j,k}u_{k,i}) \\ & + \tfrac{1}{2}B(u_{j,k}u_{k,j}u_{i,i}) + \tfrac{1}{3}C(u_{i,i}^3) + \cdots \end{aligned} \tag{3}$$

where A, B, and C are the third-order nonlinear elastic constants, μ and K are the shear and compression moduli, respectively, and where the strain tensor components are approximated as

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \tag{4}$$

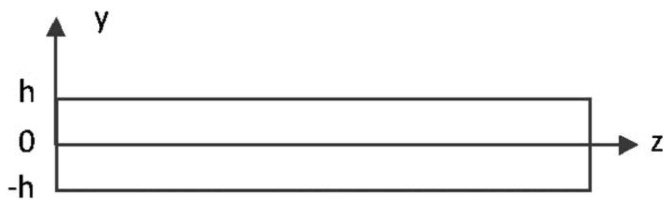

Fig. 1. Schematic of a stress free plate.

The nonlinear stress and volume forces in Eqs. (1) and (2) can be obtained from the nonlinear energy expression as [12]

$$S_{ij} = \frac{\partial E}{\partial (u_{i,j})}; \quad f_i = S_{ij,j} \tag{5}$$

3. Solution to the nonlinear problem

3.1. Forced solution to guided waves

Following Auld [10], the solution to a plate under arbitrary surface and body forces can be written as a linear combination of the existing guided wavemodes:

$$\mathbf{v}(y,z,t) = \frac{1}{2}\sum_{m=1}^{\infty} A_m(z)\mathbf{v}_m(y)\mathrm{e}^{-\mathrm{i}\omega t} \tag{6}$$

$$\mathbf{S}(y,z,t)\cdot\mathbf{n}_z = \frac{1}{2}\sum_{m=1}^{\infty} A_m(z)\mathbf{S}_m(y)\cdot\mathbf{n_z}\mathrm{e}^{-\mathrm{i}\omega t} \tag{7}$$

where $\mathbf{v} = \partial\mathbf{u}/\partial t$, $\mathbf{v}_m$ is the particle velocity of the mth mode, $\mathbf{S}_m$ is the stress tensor for the mth mode and A_m is the second-order modal amplitude to be determined. Notice that the $1/2$ is needed to ensure real quantities (de Lima and Hamilton). As shown by Auld, A_m is the solution to the following ordinary differential equation, to be solved for each individual value of m:

$$4P_{mn}\left(\frac{\mathrm{d}}{\mathrm{d}z} - \mathrm{i}\kappa_n^*\right)A_m(z) = (f_n^{\mathrm{surf}} + f_n^{\mathrm{vol}})\mathrm{e}^{\mathrm{i}\kappa z}; \quad m = 1,2,\ldots \tag{8}$$

where

$$P_{mn} = -\frac{1}{8}\int_{-h}^{h} (\mathbf{v_n^*}\cdot\mathbf{S_m} + \mathbf{v_m}\cdot\mathbf{S_n^*})\cdot\mathbf{n}_z\,\mathrm{d}\Omega \tag{9}$$

$$f_n^{\mathrm{surf}} = -\tfrac{1}{2}\mathbf{v_n^*}\mathbf{S}\cdot\mathbf{n}_y|_{y=-h}^{y=h} \tag{10}$$

$$f_n^{\mathrm{vol}} = \frac{1}{2}\int_{-h}^{h} \mathbf{f}\cdot\mathbf{v_n^*}\,\mathrm{d}y \tag{11}$$

and κ_n is the wavenumber of the wave that is not orthogonal to the mode with wavenumber κ_m. $\mathbf{S}$ and $\mathbf{f}$ are the surface traction and body force, respectively, as given by the primary wave.

3.2. Method of perturbation

de Lima and Hamilton [9] provided the solution to Eq. (8) for the case of first-order nonlinearity. By using perturbation, the solution to Eqs. (1) and (2) is written as a sum of two components:

$$\mathbf{u} = \mathbf{u^1} + \mathbf{u^2} \tag{12}$$

where $\mathbf{u^2}$ is the perturbation due to nonlinearity and is assumed to be small in comparison to $\mathbf{u^1}$. $\mathbf{u^1}$ is the solution to the following linear problem:

$$(\lambda + 2\mu)\nabla(\nabla\cdot\mathbf{u^1}) - \mu\nabla\times(\nabla\times\mathbf{u^1}) - \rho_0\frac{\partial^2\mathbf{u^1}}{\partial t^2} = 0 \tag{13}$$

$$\mathbf{S^L}(\mathbf{u^1})\cdot\mathbf{n_y} = 0 \quad \text{on } \mathscr{L} \tag{14}$$

which represents the solution to the classical linear plate problem with stress free boundary conditions. $\mathbf{u^2}$ is the solution to the forced problem:

$$(\lambda + 2\mu)\nabla(\nabla \cdot \mathbf{u^2}) - \mu\nabla \times (\nabla \times \mathbf{u^2}) - \rho_0 \frac{\partial^2 \mathbf{u^2}}{\partial t^2} = -\mathbf{f^1} \tag{15}$$

$$\mathbf{S^L}(\mathbf{u^2}) \cdot \mathbf{n_y} = -\mathbf{S}^1 \cdot \mathbf{n_y} \quad \text{on } \mathscr{L} \tag{16}$$

where $\mathbf{S^1}$ and $\mathbf{f^1}$ are surface traction and body force as calculated from the primary solution $\mathbf{u^1}$.

3.3. Solution

It must be noted that $\mathbf{S^1}$ and $\mathbf{f^1}$ are calculated by substituting the primary excitation into stress and force equations (5). If the primary excitation is a guided wavemode at a frequency ω and the energy equation (3) is nonlinear to the first order, both $\mathbf{S^1}$ and $\mathbf{f^1}$ would consequently be harmonic at 2ω. Similarily, if the energy equation also contains second-order nonlinearity, $\mathbf{S^1}$ and $\mathbf{f^1}$ would contain triple harmonic (3ω) terms.

The solution is

$$A_m(z) = \bar{A}_m(z)e^{i(2\kappa z)} - \bar{A}_m(0)e^{i\kappa_n^* z} \tag{17}$$

where

$$\bar{A}_m(z) = i\frac{(f_n^{\text{vol}} + f_n^{\text{surf}})}{4P_{mn}[\kappa_n^* - 2\kappa]}; \quad \kappa_n^* \neq 2\kappa \tag{18}$$

$$\bar{A}_m(z) = \frac{(f_n^{\text{vol}} + f_n^{\text{surf}})}{4P_{mn}} z; \quad \kappa_n^* = 2\kappa \tag{19}$$

where A_m are the amplitudes of the modes at 2ω and κ is the wavenumber of the primary wave.

4. Condition for the absence of antisymmetric modes

The first-order nonlinear solution to the guided wave problem (Eq. (17)) can be extended to higher orders by using appropriates $\mathbf{S}$ and $\mathbf{f}$ in Eqs. (10) and (11) and by using normal mode expansion at the appropriate higher harmonic. Since the method of perturbation reduces the nonlinear problem to a forced linear problem, we can, for the sake of simplicity, assume that the energy expression (Eq. (3)) consists of any one single order of nonlinearity. Consequently, $\mathbf{S}$ and $\mathbf{f}$ are due to that particular order of nonlinearity alone. Furthermore, for the sake of convenience, we will follow the convention wherein Q^{Ay} and Q^{Sy} represents that any quantity, Q, is antisymmetric and symmetric w.r.t. the thickness direction (y direction), respectively.

It can be seen from Eqs. (18) and (19) that a particular mode would not be excited at a higher harmonic if both f_n^{surf} and f_n^{vol} equal zero. For a higher harmonic antisymmetric driven Rayleigh–Lamb mode, $\mathbf{v}_n$ in Eq. (9), we have

$$\mathbf{v}_{ny} = v_y^{Sy}; \quad \mathbf{v}_{nz} = v_z^{Ay} \tag{20}$$

From Eq. (10), it can be seen that f_n^{surf} would be zero if the quantity $\mathbf{v_n^*S} \cdot \mathbf{n}_y$ is symmetric w.r.t. y. For this to be true, from Eq. (20), we find that $\mathbf{S}$ should be of the following form:

$$\mathbf{S} = \begin{bmatrix} S_{yy}^{Sy} & S_{yz}^{Ay} \\ S_{zy}^{Ay} & S_{zz}^{Sy} \end{bmatrix} \tag{21}$$

Also from Eq. (11), it can be seen that f_n^{vol} would be zero if the quantity $\mathbf{f} \cdot \mathbf{v_n^*}$ is antisymmetric w.r.t. y (due to integration over y). For this to be true, from Eq. (20), we find that $\mathbf{f}$ should be of the following form:

$$\mathbf{f} = [f_y^{Ay} \quad f_z^{Sy}] \tag{22}$$

It is worth noting here that a derivative w.r.t. y makes a symmetric function antisymmetric and vice versa, whereas a derivative w.r.t. z does not affect the symmetry of the function. In other words, if Q^{Ay}, then $Q^{Sy}_{,y}$, etc. It follows from the previous logic and $f_i = S_{ij,j}$ that if $\mathbf{S}$ is of the form shown in Eq. (21), then $\mathbf{f}$ would automatically be of the form shown in Eq. (22). It should also be noted that $\mathbf{f}$ can be of the form shown in Eq. (22) only if $\mathbf{S}$ is of the form shown in Eq. (21). Since one implies the other, it is necessary to satisfy only one of the two conditions.

From Eq. (5), we have the following:

$$\begin{aligned} f_i &= \left[\frac{\partial E}{\partial u_{i,j}}\right]_{,j} \\ &= \left[\frac{\partial}{\partial x_l}\frac{\partial x_l}{\partial u_{i,j}}(E)\right]_{,j} \\ &= \left[E_{,l}\frac{1}{u_{i,jl}}\right]_{,j} \\ &= \left[E_{,y}\frac{1}{u_{i,yy}} + E_{,z}\frac{1}{u_{i,yz}}\right]_{,y} + \left[E_{,y}\frac{1}{u_{i,zy}} + E_{,z}\frac{1}{u_{i,zz}}\right]_{,z} \end{aligned} \tag{23}$$

for a plain strain, Rayleigh–Lamb mode.

From the above equation, it can be verified that Eqs. (21) and (22) would hold if the following conditions are satisfied:

Absence of antisymmetric harmonics:

$$\begin{aligned} &E^{Sy} \text{ when } u_y^{Ay} \text{ and } u_z^{Sy} \text{ (symmetric driving mode)} \\ &E^{Ay} \text{ when } u_y^{Sy} \text{ and } u_z^{Ay} \text{ (antisymmetric driving mode)} \end{aligned} \tag{24}$$

In other words, the symmetry of E follows the symmetry of $u_{l,l}$ (for either an antisymmetric or a symmetric driving mode). This observation will make subsequent proofs simpler.

Similarly, a set of conditions necessary and sufficient for the absence of symmetric motion ($v_{ny} = v_y^{Ay}, v_{nz} = v_z^{Sy}$) becomes

Absence of symmetric harmonics:

$$\begin{aligned} &E^{Ay} \text{ when } u_y^{Ay} \text{ and } u_z^{Sy} \text{ (symmetric driving mode)} \\ &E^{Sy} \text{ when } u_y^{Sy} \text{ and } u_z^{Ay} \text{ (antisymmetric driving mode)} \end{aligned} \tag{25}$$

To summarize, no symmetric harmonics are allowed when the symmetry of E is orthogonal to the symmetry of $u_{l,l}$ (for either a symmetric or an antisymmetric driving mode).

It follows from conditions (Eqs. (24) and (25)) that a given harmonic can allow both symmetric and antisymmetric motions under the following conditions:

$$\begin{aligned} &E^{Sy} \text{ when } u_y^{Ay} \text{ and } u_z^{Sy} \text{ (symmetric driving mode)} \\ &E^{Sy} \text{ when } u_y^{Sy} \text{ and } u_z^{Ay} \text{ (antisymmetric driving mode)} \end{aligned} \tag{26}$$

In other words, E is always symmetric.

5. Application to first-order nonlinearity

Energy relation for first-order nonlinearity (from Eq. (3)), assuming compatibility relation, Eq. (4), contains terms which are cubic in displacement derivatives. The relation, with some rearrangements, is reproduced here for the sake of clarity:

$$E = E_1(u_{i,j}u_{j,k}u_{k,i}) + E_2(u_{i,j}u_{k,i}u_{k,j}) + E_3(u_{i,i}u_{j,k}u_{k,j}) + E_4[u_{i,i}(u_{j,k})^2] + E_5(u_{i,i}^3) \tag{27}$$

where E_1, E_2, E_3, E_4, and E_5 are constants.

From Eq. (24), it is enough to show that if each of the five terms in the above equation follows the symmetry of $u_{l,l}$, first-order nonlinearity would not support any antisymmetric higher order modes. In other words, we need to show that every term in Eq. (27) can be written as $u_{l,l} \times Q^{Sy}$, where Q^{Sy} is a symmetric function.

Before proceeding with the proof, the following should be noted:

$$u_{i,j}u_{j,i} = u_{y,y}u_{y,y} + u_{y,z}u_{z,y} + u_{z,y}u_{y,z} + u_{z,z}u_{z,z} \tag{28}$$

It can be verified that all the terms in the above expansion are symmetric irrespective of the symmetry of the mode in question, hence:

Lemma 1. *All the terms in the expansion of* $u_{i,j}u_{j,i}$ *are always symmetric, hence* $[u_{i,j}u_{j,i}]^{Sy}$.

Also,

$$u_{i,i}u_{j,j} = (u_{y,y} + u_{z,z})^2 \tag{29}$$

Hence:

Lemma 2. $[u_{i,i}u_{j,j}]^{Sy}$.

Further,

$$u_{i,j}u_{i,j} = (u_{i,j})^2 \tag{30}$$

Hence:

Lemma 3. $[u_{i,j}u_{i,j}]^{Sy}$.

The energy expression for first-order nonlinearity contains terms which have cubic powers of displacement derivatives. Denoting the first term in Eq. (27) (without the constant multiplier) as $T = u_{i,j}u_{j,k}u_{k,i}$, we have the following cases:

(a) $i = j = p$; $T = u_{\bar{p},\bar{p}}u_{\bar{p},k}u_{k,\bar{p}}$, where an overbar indicates that the corresponding index does not follow Einstein convention (since p is either y or z). From Lemma (1), $u_{\bar{p},k}u_{k,\bar{p}} = Q^{Sy}$. Hence, $T = u_{\bar{p},\bar{p}}Q^{Sy}$.
(b) $i \neq j$; since i, j and k can assume only y and z, and since $i \neq j$, the third index, k, must be equal to either i or j. When $k = i = p$, $T = u_{\bar{p},j}u_{j,\bar{p}}u_{\bar{p},\bar{p}}$. Conversely, when $k = j = p$, $T = u_{i,\bar{p}}u_{\bar{p},\bar{p}}u_{\bar{p},i}$. In either case, from Lemma 1, it follows that $T = u_{\bar{p},\bar{p}}Q^{Sy}$.

For the second term in Eq. (27), $T = u_{i,j}u_{k,i}u_{k,j}$, the following two cases arise:

(a) $i = j = p$; $T = u_{\bar{p},\bar{p}}u_{k,\bar{p}}u_{k,\bar{p}}$. From Lemma 2, $T = u_{\bar{p},\bar{p}}Q^{Sy}$.
(b) $i \neq j$. Since i, j and k can assume only y and z, and since $i \neq j$, the third index, k, must be equal to either i or j. When $k = i = p$, $T = u_{\bar{p},j}u_{\bar{p},\bar{p}}u_{\bar{p},j}$. Conversely, when $k = j = p$, $T = u_{i,\bar{p}}u_{\bar{p},i}u_{\bar{p},\bar{p}}$. In either case, from Lemmas 1 and 3, it follows that $T = u_{\bar{p},\bar{p}}Q^{Sy}$.

It can be seen that the other terms in Eq. (27) ($u_{i,k}u_{k,i}u_{l,l}, u_{l,l}(u_{i,k})^2, u_{l,l}^3$) trivially reduce to $T = u_{i,i}Q^{Sy}$ by the applications of Lemmas 1–3. Hence, from conditions (24), first-order nonlinearity under hyper-elasticity model cannot support antisymmetric Rayleigh–Lamb waves.

6. Application to higher order harmonics

Subsequent proofs to harmonics of order higher than two will be presented by mathematical induction. For higher order nonlinearity, higher order compatibility relations should be used [13]. However, in order to draw conclusions on a certain order of higher harmonic, it is sufficient to examine the corresponding energy expression in terms of powers of strains (since an $E(u_{i,j})^n$ term will generate the $(n-1)$ order harmonic).

Theorem 4. *If the energy relation for an nth order harmonic,* $E^n = u_{l,l}Q^{Sy}$, *n is even and for the next higher order,* $E^{n+1} = P^{Sy}$, *where P and Q are symmetric functions.*

For an nth order harmonic, all the terms in the energy expression contain $n+1$ powers of strains. $E^n = u_{l,l}Q^{Sy}$ implies any arbitrary term in the expansion of E^n (denoted by T^n) which behaves like $u_{l,l}Q^{Sy}$. By 'any arbitrary term', we mean all $T^n = F^{n+1}(i_1, i_2, \ldots, i_{n+1})$, where $F^{n+1}(i_1, i_2, \ldots, i_{n+1})$ represents a function having $n+1$ multiples of strains which depend upon $n+1$ indices (i_1–i_{n+1}) and where every index occurs exactly twice so that Einstein summation convention applies to all the indices.

Saying that all such $T^n = u_{l,l}Q^{Sy}$ also mean that n is an even number because if n was an odd number, at least one term in the expansion of E would equal $(u_{l,l})^{n+1}$ which, given the assumption that n is odd, would always be symmetric. Hence, our initial assumption that all $T^n = u_{l,l}Q^{Sy}$ would become false.

Following two scenarios arise:

Lemma 5. *If* T^{n+1} *contains at least one strain term with repeated index,* u_{i_m,i_m}:

$$\begin{aligned} T^{n+1} &= u_{i_m,i_m}F^{n+1}(i_1, i_2, \ldots, i_{m-1}, i_{m+1}, \ldots, i_{n+2}) \\ &= u_{i_m,i_m}F^{n+1}(i'_1, i'_2, \ldots, i'_{n+1}) \\ &= u_{i_m,i_m}T^n \\ &= u_{i_m,i_m}u_{l,l}Q^{Sy} \\ &= P^{Sy} \end{aligned}$$

Lemma 6. *If* T^{n+1} *contains at least one strain term where the two indices assume equal values, or* $i_j = i_k = p(p = y$ *or* $z)$:

$$\begin{aligned} T^{n+1} &= u_{\bar{p},\bar{p}}F^{n+1}(i_1, i_2, \ldots, (i_j = \bar{p}), \ldots, (i_k = \bar{p}), \ldots, i_{n+2}) \\ &= u_{\bar{p},\bar{p}}F^{n+1}(i'_1, i'_2, \ldots, i'_{n+1})(i_j, i_k = i'_n = \bar{p}) \\ &= u_{\bar{p},\bar{p}}T^n \\ &= u_{\bar{p},\bar{p}}u_{l,l}Q^{Sy} \\ &= P^{Sy} \end{aligned}$$

Moving forward, the most generic energy term for $(n+1)$th order harmonic can be expressed as

$$T^{n+1} = u_{i_1,i_2}F^{n+1}(i_1, i_2, \ldots, i_{n+2})$$

The following two cases arise:

(a) $i_1 = i_2 = p$: In this case $T^{n+1} = P^{Sy}$ from Lemma 6.

(b) $i_1 \neq i_2$; since each index from i_1 to i_{n+2} equals either y or z and since $i_1 \neq i_2$, each of i_3–i_{n+2} equals either i_1 or i_2. Keeping in mind Lemmas 5 and 6, we are only concerned with the nontrivial case where all the terms T^{n+1} have indices which assume different values. In other words, every term in T^{n+1} is either u_{i_1,i_2} or u_{i_2,i_1}. Since $(n+2)$ is even, we can divide T^{n+1} into $(n+2)/2$ multiplied sets of multiples of 2 terms each. Each such set is either $u_{i_1,i_2}u_{i_2,i_1}$ or $(u_{i_1,i_2})^2$ or $(u_{i_2,i_1})^2$. All of these (Lemmas 1–3) are symmetric, hence their product is also symmetric, hence $T^{n+1} = P^{Sy}$.

This completes the proof.

Theorem 7. *If the energy relation for an nth order harmonic,* $E^n = Q^{Sy}$, *n is odd and for the next higher order,* $E^{n+1} = u_{l,l}P^{Sy}$, *where P and Q are symmetric functions.*

Saying that all $T^n = Q^{Sy}$ mean that n is an odd number because if n was an even number, at least one term in the expansion of E would equal $(u_{l,l})^{n+1}$ which, given the assumption that n is even, would become $u_{l,l}Q^{Sy}$. Hence, our initial assumption that all $T^n = Q^{Sy}$ would become false.

Following two scenarios arise:

Lemma 8. *If T^{n+1} contains at least one strain term with repeated index, u_{i_m,i_m}:*

$$\begin{aligned} T^{n+1} &= u_{i_m,i_m}F^{n+1}(i_1, i_2, \ldots, i_{m-1}, i_{m+1}, \ldots, i_{n+2}) \\ &= u_{i_m,i_m}F^{n+1}(i'_1, i'_2, \ldots, i'_{n+1}) \\ &= u_{i_m,i_m}T^n \\ &= u_{i_m,i_m}P^{Sy} \end{aligned}$$

Lemma 9. *If T^{n+1} contains at least one strain term where the two indices assume equal values, or $i_j = i_k = p(p = y$ or $z)$:*

$$\begin{aligned} T^{n+1} &= u_{\bar{p},\bar{p}}F^{n+1}(i_1, i_2, \ldots, (i_j = \bar{p}), \ldots, (i_k = \bar{p}), \ldots, i_{n+2}) \\ &= u_{\bar{p},\bar{p}}F^{n+2}(i'_1, i'_2, \ldots, i'_{n+1})(i_j, i_k = i'_n = \bar{p}) \\ &= u_{\bar{p},\bar{p}}T^n \\ &= u_{\bar{p},\bar{p}}P^{Sy} \end{aligned}$$

Moving forward, the most generic energy term for $(n+1)$th order harmonic can be expressed as

$$T^{n+1} = u_{i_1,i_2}F^{n+1}(i_1, i_2, \ldots, i_{n+2})$$

The following two cases arise:

(a) $i_1 = i_2 = p$: in this case, $T^{n+1} = u_{\bar{p},\bar{p}}P^{Sy}$ from Lemma 9.

(b) $i_1 \neq i_2$; since each index from i_1 to i_{n+2} equals either y or z and since $i_1 \neq i_2$, each of i_3–i_{n+2} equals either i_1 or i_2. Keeping in mind Lemmas 8 and 9, we want to see if it is even possible to have T^{n+1} where all the terms have indices which assume different values. Since n is odd, the total number of indices $n+2$ is odd. To have a scenario where there is no index repetition and where every multiple in T^{n+1} is u_{i_1,i_2} or u_{i_2,i_1}, the number of indices assuming value i_1 should be equal to the number of indices assuming value i_2. Since the total number of indices in the present case is odd, we have at least the following sets of indices:
   1. $i'_1, i'_2, \ldots, i'_{(n+1)/2}$ with each being equal to i_1.
   2. $i''_1, i''_2, \ldots, i''_{(n+1)/2}$ with each being equal to i_2.
   3. i_m equal to either i_1 or i_2.

It should be seen that the total number of indices in the above three sets is equal to $n+2$. It can be shown that, from the first two sets, $n+1$ displacement derivatives constituting a term in nth order harmonic $F^{n+1}(i'_1, \ldots, i'_{(n+1)/2}, i''_1, \ldots, i''_{(n+1)/2})$ is obtained by using each index exactly twice. The last $(n+2)$th term is thus left with just one index to utilize since all the other indices have already been used twice. Hence, that last term has the form u_{i_m,i_m}. Hence,

$$\begin{aligned} T^{n+1} &= u_{i_m,i_m}F^{n+1}(i'''_1, i'''_2, \ldots, i'''_{n+1}) \\ &= u_{i_m,i_m}T^n \\ &= u_{i_m,i_m}P^{Sy} \\ &= u_{l,l}P^{Sy} \end{aligned}$$

This completes the proof.

From Theorems (4 and 7) and conditions (24) and (26), all even harmonics (2ω, 4ω, 6ω, ...) support only symmetric Rayleigh–Lamb waves, whereas all odd harmonics (3ω, 5ω, 7ω, ...) support both symmetric and antisymmetric Rayleigh–Lamb waves.

7. Experimental confirmation

Two experiments were carried out to test the theoretical result. In the first experiment, one Pico transducer (Physical Acoustics Corporation, 0.1–1 MHz, central frequency 0.543 MHz) was used to generate Lamb waves in an aluminum plate of thickness 2.54 mm. The response was measured at a distance of 25 cm by a Pinducer sensor (Valpey Fisher VP-1093). The plate was loaded quasi-statiscally to a level large enough to induce measurable nonlinearity driven higher harmonics of the primary Rayleigh–Lamb wave. Both the Pico and the Pinducer work by exciting and sensing out of plane displacements, therefore, generate and receive predominantly antisymmetric motion. The excitation was driven at a monochromatic frequency of 320 kHz. Fig. 2(b) shows frequency content of the received signal.

Since it is the antisymmetric motion that is predominantly received, it is expected that the even harmonics do not figure prominently in the result. This can be verified in Fig. 2 where only the odd harmonics are distinguishable.

As a further confirmation of the concept in the experimental results, a joint time–frequency analysis of the received signal was conducted (complex Morlet wavelet, F_b (bandwidth parameter) = 2, F_c (central frequency parameter) = 2.5). Fig. 3(b) shows the wavelet scalogram applied to the time signal depicted in Fig. 3(a). Fig 3(c) shows a zoomed in view of the scalogram in the frequency range of 0.5–1 MHz. The white lines are the theoretical arrival times of the pertinent modes according to Rayleigh–Lamb theory. It can be seen that while a strong antisymmetric mode is present at the generation frequency (320 kHz), there is no antisymmetric mode present at the double harmonic (640 kHz). The antisymmetric mode is, instead, present at the triple harmonic (960 kHz), as predicted by the theoretical formulations and identified in the frequency spectrum of Fig. 2.

In the second experiment, two macro-fiber composite (MFC) transducers (Smart Materials Corporation, M2814P1) were used for both excitation and detection. MFCs work by generating and detecting in-plane strains and hence they are preferentially sensitive to symmetric waves. Fig. 4(b) shows the frequency content of the received signal.

The primary generation frequency was, again, 320 kHz. Since the sensitivity of the transducers to symmetric waves is higher in this experiment, we expect a better representation of even harmonics. As seen from Fig. 4(b), the double harmonic (640 kHz) emerges from the spectrum. Even harmonics higher than the double are too weak to be detected. The large odd harmonic at 960 kHz may be attributed to two factors. The first reason is

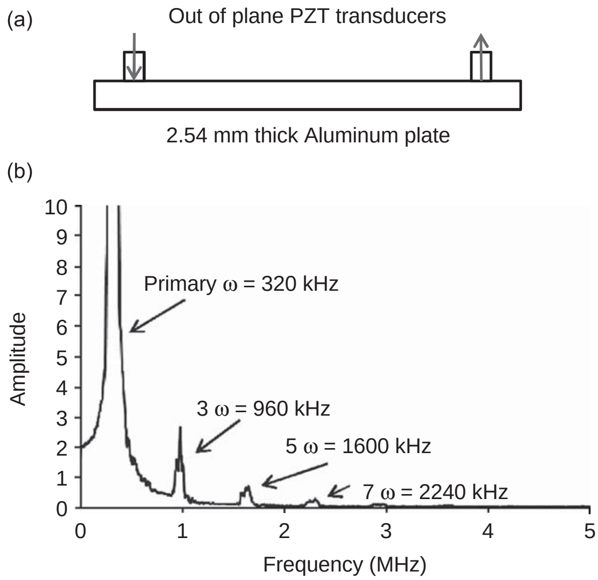

Fig. 2. Measurement of nonlinear higher harmonics: (a) schematic of the experiment and (b) frequency content of the signal received by the Pinducer.

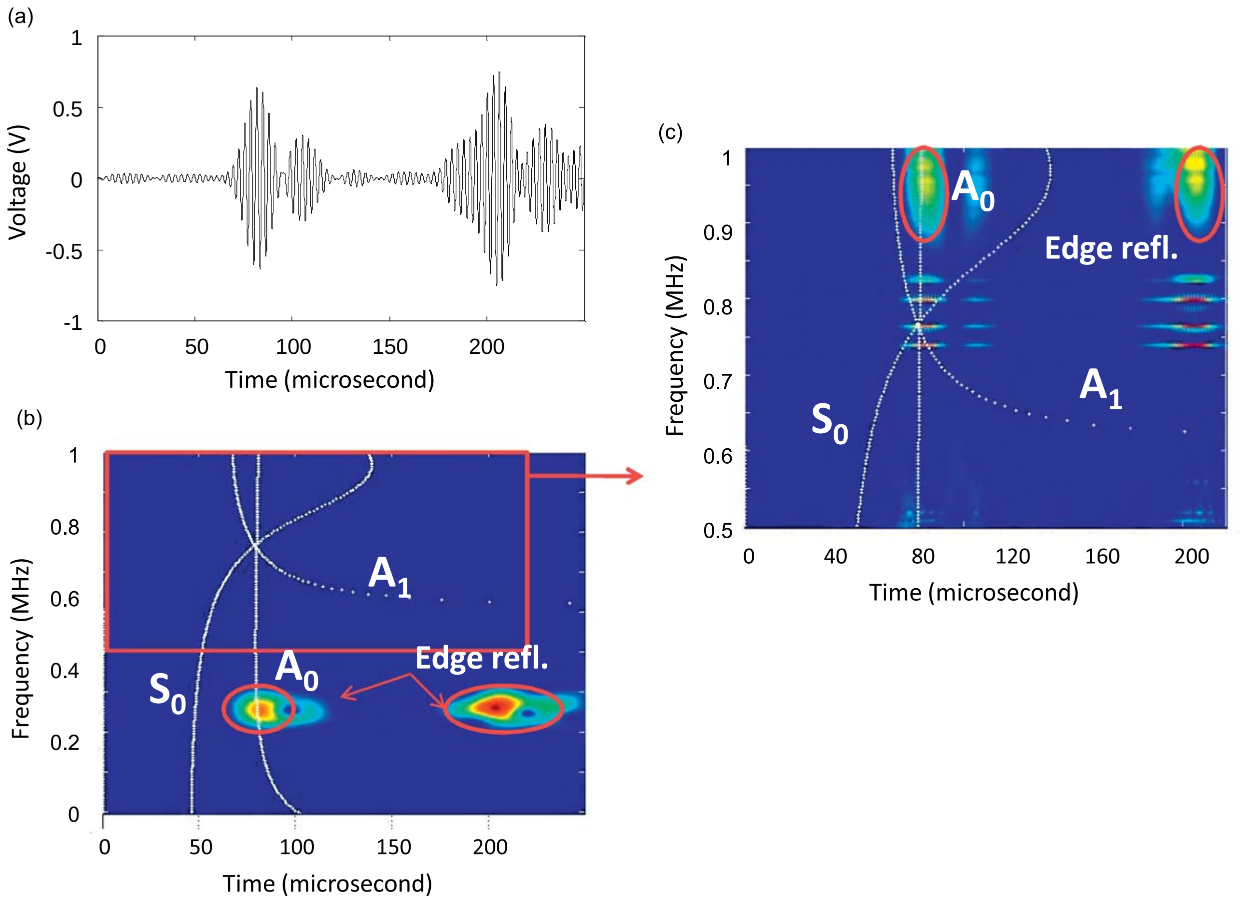

Fig. 3. Joint time–frequency analysis of the antisymmetric excitation and detection in the plate: (a) time history, (b) wavelet scalogram of the signal in the DC-1 MHz range, (c) zoomed view of the wavelet scalogram. White lines are the theoretical arrival times from the Rayleigh–Lamb formulation.

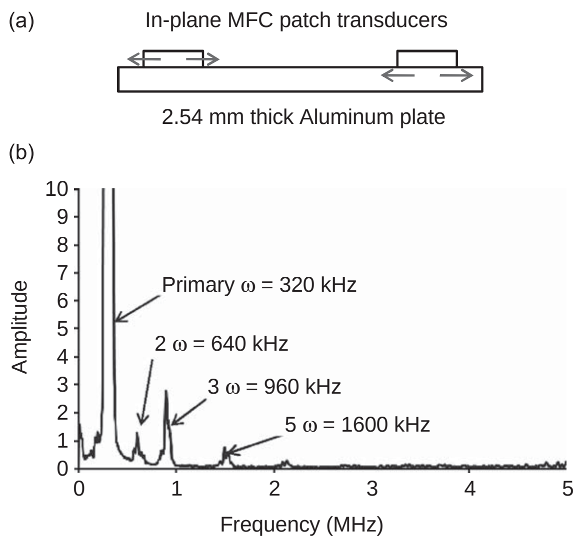

Fig. 4. Measurement of nonlinear higher harmonics: (a) schematic of the experiment and (b) frequency content of the signal received by the MFC patch.

that the odd harmonic consists of both symmetric and antisymmetric motion (the antisymmetric contribution due to the fact that MFC patches are not "pure mode" transducers). The second reason is the presence of harmonic contributions in the undeformed plate (experimental contributions). Fig. 5 shows the comparison of the frequency content of signals in both unloaded plate and loaded plate configurations. It can be seen that

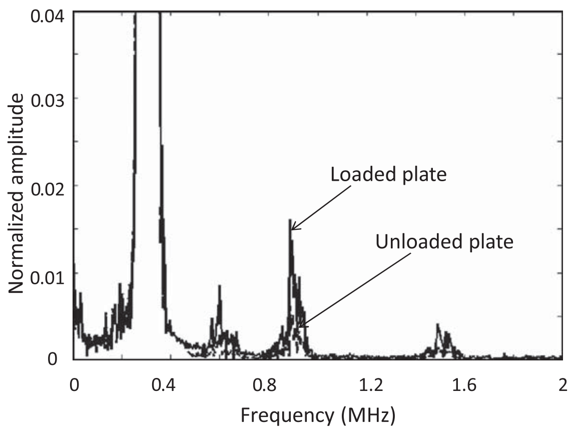

Fig. 5. Frequency content comparison of signals in unloaded and loaded cases.

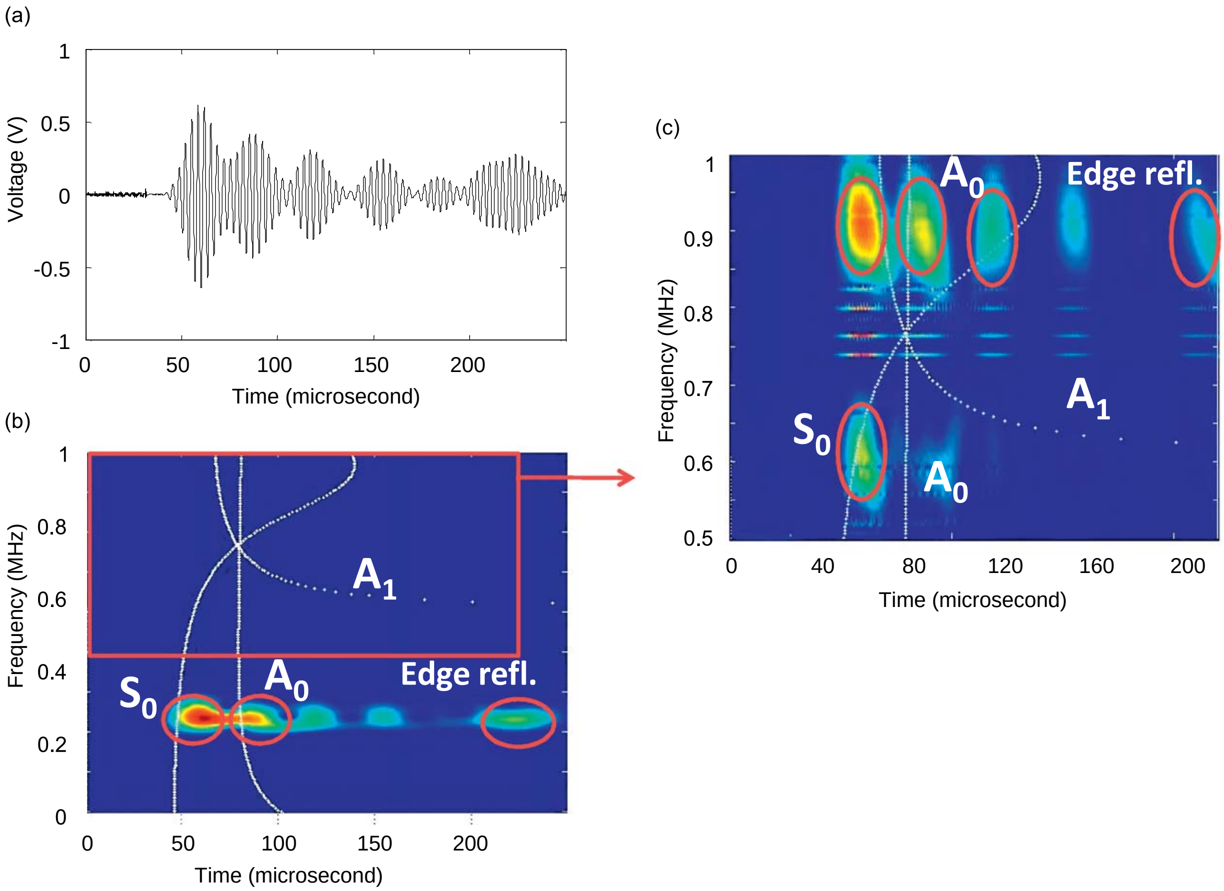

Fig. 6. Joint time–frequency analysis of the symmetric excitation and detection in the plate: (a) time history, (b) wavelet scalogram, (c) zoomed view of the wavelet scalogram. White lines are theoretical arrival times from Rayleigh–Lamb formulation.

although harmonic contributions are present in the unloaded plate, their magnitude is significantly smaller compared to the case of a fully loaded plate. It can thus be concluded that, indeed, a substantial portion of the measured higher harmonic is due to nonlinear elasticity of the plate rather than to experimental conditions.

Fig. 6 shows the continuous wavelet scalogram of the received signal in the antisymmetric experiment, along with the theoretical Rayleigh–Lamb curves of the pertinent modes. It can be seen from Fig. 6(b) that while the primary harmonic (320 kHz) consists of both symmetric and antisymmetric modes, the energy at the double harmonic (640 kHz) corresponds exclusively to the fundamental symmetric mode (Fig. 6c). The energy at the triple harmonic (960 kHz), as expected, consists of a combination of the fundamental antisymmetric, the fundamental symmetric, and the first-order antisymmetric modes. Hence the experiments confirm that the antisymmetric modes can only exist at odd harmonics, whereas symmetric modes can exist at both odd and even harmonics.

8. Conclusions

The nonlinear Rayleigh–Lamb guided wave problem was studied using the method of perturbation coupled with wavemode orthogonality and forced response. It was found that the inability of an even harmonic in supporting antisymmetric motion results from constraints on the corresponding energy equation. These conditions were derived and they were used to explain why the double harmonic does not allow antisymmetric Rayleigh–Lamb waves. Principles of mathematical induction, used to generalize to higher order of harmonics, concluded that antisymmetric Rayleigh–Lamb waves are only allowed at odd harmonics, whereas symmetric Rayleigh–Lamb waves are allowed at all (odd or even) harmonics.

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